MAPPING CYLINDERS AND THE OKA PRINCIPLE

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Abstract. We apply concepts and tools from abstract homotopy theory to complex analysis and geometry, continuing our development of the idea that the Oka Principle is about fibrancy in suitable model structures. We explicitly factor a holomorphic map between Stein manifolds through mapping cylinders in three different model structures and use these factorizations to prove implications between ostensibly different Oka properties of complex manifolds and holomorphic maps. We show that for Stein manifolds, several Oka properties coincide and are characterized by the geometric condition of ellipticity. Going beyond the Stein case to a study of cofibrant models of arbitrary complex manifolds, using the Jouanolou Trick, we obtain a geometric characterization of an Oka property for a large class of manifolds, extending our result for Stein manifolds. Finally, we prove a converse Oka Principle saying that certain notions of cofibrancy for manifolds are equivalent to being Stein.

Introduction. In this paper, we apply concepts and tools from abstract homotopy theory to complex analysis and geometry, based on the foundational work in [L2], continuing our development of the idea that the Oka Principle is about fibrancy in suitable model structures. A mapping cylinder in a model category is an object through which a given map can be factored as a cofibration followed by an acyclic fibration (or sometimes merely an acyclic map). We explicitly factor a holomorphic map between Stein manifolds through mapping cylinders in three different model structures. We apply these factorizations to the Oka Principle, mainly to prove implications between ostensibly different Oka properties of complex manifolds and holomorphic maps. We show that for Stein manifolds, several Oka properties coincide and are characterized by the geometric condition of ellipticity. We then move beyond the Stein case to a study of cofibrant models of arbitrary complex manifolds (this involves the same sort of factorization through a mapping cylinder as before). Using the so-called Jouanolou Trick, we obtain a geometric characterization of an Oka property for a large class of manifolds, extending our result for Stein manifolds. Finally, we prove a "converse Oka Principle" saying that certain notions of cofibrancy for manifolds are

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equivalent to being Stein. This confirms and makes precise the impression that the Stein property is dual to the Oka properties that have been proved equivalent to fibrancy.

Factorization in the intermediate structure. We shall work in the simplicial category $\mathfrak{S}$ of prestacks on the simplicial site $\mathcal{S}$ of Stein manifolds. The category of complex manifolds is simplicially embedded in $\mathfrak{S}$. There are at least six interesting simplicial model structures on $\mathfrak{S}$: the projective, intermediate, and injective structures with respect to either the trivial topology on $\mathcal{S}$ or the topology in which a cover, roughly speaking, is a cover by Stein open subsets. These structures may be used to study the Oka Principle for complex manifolds and holomorphic maps. We sometimes refer to notions associated with the former topology as *coarse* and the latter as *fine*. We often omit the word *fine*, as the fine structures are more important than the coarse ones. For more background, see [L2], where the intermediate structures were first introduced, and the references there.

One of the axioms for a model category states that any map can be factored (by no means uniquely) as a cofibration followed by an acyclic fibration. The general verification of this for a category of prestacks is quite abstract and not always useful in practice. We start by producing an explicit factorization in the intermediate structures — they have the same cofibrations and acyclic fibrations — for a holomorphic map $f : R \to S$ between Stein manifolds. We assume that R is finite-dimensional, meaning that the (finite) dimensions of its connected components are bounded (by convention, we take manifolds to be second countable but not necessarily connected).

Let $\phi : R \to \mathbb{C}^r$ be a holomorphic embedding of R into Euclidean space. Let M be the Stein manifold $S \times \mathbb{C}^r$. Then f can be factored as $R \xrightarrow{j} M \xrightarrow{\pi} S$, where $j = (f, \phi)$ is an embedding and π is the projection $(x, z) \mapsto x$. In the language of [L2], j is a Stein inclusion, so it is an intermediate cofibration. Since $\mathbb{C}^r$ is elliptic, π is an intermediate fibration: it is the pullback of the constant map from $\mathbb{C}^r$ by the constant map from S. Also, π is pointwise acyclic since $\mathbb{C}^r$ is holomorphically contractible. Following standard terminology in topology, we refer to M as a *mapping cylinder* for f.

Interpolation implies approximation. Complex manifolds and, as introduced in [L2], holomorphic maps can possess various Oka properties. First, there are the parametric Oka property (POP) and the strictly stronger parametric Oka property with interpolation on a submanifold (POPI). For manifolds and those holomorphic maps that are topological fibrations (in the sense of Serre or Hurewicz: these are equivalent for continuous maps between smooth manifolds), POP, called the weak Oka property in [L2], is equivalent to projective fibrancy, and POPI, called *the* Oka property in [L2], is equivalent to intermediate fibrancy.

There is also the parametric Oka property with approximation on a holomorphically convex compact subset (POPA): this property has not been explicitly generalized from manifolds to maps until now; the natural notion appears below. Each parametric property has a *basic* version (BOP, BOPI, BOPA), in which the parameter space is simply a point. A disc satisfies POP but neither BOPI nor BOPA. For more background, see [L2] and the survey [F2].

Approximation on holomorphically convex compact subsets is important in most areas of complex analysis, including basic proofs in Oka theory. It is therefore of interest that POPI implies POPA. The following argument is short and easy and makes use of the mapping cylinder introduced above. It seems reasonable to view POPA as a reflection of a special case of POPI, but not as a possibly distinct notion of intermediate fibrancy in its own right. We do not know whether POPA is strictly weaker than POPI: examples in this area are hard to come by.

Let $k : X \to Y$ be a holomorphic map satisfying POPI. Let P be a finite polyhedron with a subpolyhedron Q. Let S be a Stein manifold and K be a holomorphically convex compact subset of S with a neighbourhood U. Let $h : S \times P \to X$ be a continuous map such that

(a) $h(\cdot, p)$ is holomorphic on U for all $p \in P$,
(b) $h(\cdot, q)$ is holomorphic on S for all $q \in Q$, and
(c) $k \circ h(\cdot, p)$ is holomorphic on S for all $p \in P$.

Let d be a metric on X compatible with the topology of X and let $\epsilon > 0$. To show that k satisfies POPA we need to find a continuous deformation $H : S \times P \times I \to X$, where I is the unit interval $[0, 1]$, such that

(1) $H(\cdot, \cdot, 0) = h$,
(2) $H(\cdot, p, 1)$ is holomorphic on S for all $p \in P$,
(3) $H(\cdot, q, t) = h(\cdot, q)$ for all $q \in Q$ and $t \in I$,
(4) $k \circ H(\cdot, p, t) = k \circ h(\cdot, p)$ all $p \in P$ and $t \in I$, and
(5) $d(H(x, p, t), h(x, p)) < \epsilon$ for all $(x, p, t) \in K \times P \times I$.

Let $R \subset U$ be a finite-dimensional Stein neighbourhood of K. The inclusion $f : R \to S$ factors through the mapping cylinder $M = S \times \mathbb{C}^r$ introduced above as $R \xrightarrow{j} M \xrightarrow{\pi} S$. Define $g = h(\pi(\cdot), \cdot) : M \times P \to X$. Then g is continuous; $g(\cdot, p)$ is holomorphic on R (more precisely, $g(j(\cdot), p)$ is holomorphic) for all $p \in P$; $g(\cdot, q)$ is holomorphic on M for all $q \in Q$; and $k \circ g(\cdot, p)$ is holomorphic on M for all $p \in P$. Since k satisfies POPI, there is a continuous deformation $G : M \times P \times I \to X$ such that $G(\cdot, \cdot, 0) = g$; $G(\cdot, p, 1)$ is holomorphic on M for all $p \in P$; $G(\cdot, q, t) = g(\cdot, q)$ for all $q \in Q$ and $t \in I$; and finally $G(j(\cdot), p, t) = g(j(\cdot), p)$ and $k \circ G(\cdot, p, t) = k \circ g(\cdot, p)$ for all $p \in P$ and $t \in I$.

Since K is holomorphically convex in S, the embedding $\phi : R \to \mathbb{C}^r$ can be uniformly approximated on K as closely as we wish by a holomorphic map $\psi : S \to \mathbb{C}^r$. Precomposing the deformation G in its first argument by the holomorphic section $\sigma = (\mathrm{id}_S, \psi) : S \to M$ of π, we obtain a continuous map $H : S \times P \times I \to X$ such that

$$H(\cdot, \cdot, 0) = G(\sigma(\cdot), \cdot, 0) = g(\sigma(\cdot), \cdot) = h(\pi \circ \sigma(\cdot), \cdot) = h,$$

so (1) holds. Verifying (2), (3), and (4) is straightforward. Finally, for (x, p, t) in the compact space $K \times P \times I$,

$$\begin{aligned} d(H(x, p, t), h(x, p)) &= d(G(\sigma(x), p, t), g(j(x), p)) \\ &= d(G((x, \psi(x)), p, t), G((x, \phi(x)), p, t)) \end{aligned}$$

is as small as we like. Let us record what we have just proved.

Theorem 1. *For holomorphic maps between complex manifolds, the parametric Oka property with interpolation implies the parametric Oka property with approximation.*

Taking Y to be a point shows that POPI implies POPA for manifolds. Taking Q to be empty and P to be a point shows that BOPI implies BOPA. It also follows that an Oka manifold or a manifold merely satisfying BOPI has a dominating holomorphic map from Euclidean space; this is not obvious from the definition of BOPI.

Basic interpolation implies ellipticity for Stein manifolds. Ellipticity, introduced by Gromov [G], and the more general subellipticity, introduced by Forstnerič [F1], are geometric sufficient conditions for all Oka properties, including one we have not mentioned before: the parametric Oka property with *jet* interpolation on a submanifold (POPJI), which involves fixing not just the values but a finite-order jet of a map along a submanifold as the map is deformed to a globally holomorphic map. Gromov noted that for a Stein manifold, the basic version of this property implies ellipticity [G, 3.2.A]. With the help of a mapping cylinder and a more sophisticated Runge approximation than above we will now show that ellipticity of a Stein manifold actually follows from the basic Oka property with plain, zeroth-order interpolation.

Let X be a Stein manifold. Without loss of generality we may assume that X is connected. The proof of Gromov's result given by Forstnerič and Prezelj [FP, Proposition 1.2] produces a smooth map u from the tangent bundle $Y = TX$ of X, which is a Stein manifold, to X, such that u is the identity on the zero section Z of TX (when Z is identified with X), u is holomorphic on a neighbourhood V of Z, and the derivative of u at a point of Z restricts to the identity map of the tangent space of X at that point, viewed as a subspace of the tangent space of Y there. If u was holomorphic on all of Y, it would be a dominating spray and X would be elliptic. The basic Oka property with jet interpolation on Z allows us to deform u to a dominating spray without further ado; we will produce a dominating family of sprays on X using only plain interpolation.

Every analytic subset of a Stein manifold has a fundamental system of Runge neighbourhoods [KK, E. 63e], so we may assume that V is Runge (as many but not all authors do, we take Runge to include Stein). Let $\phi : V \to \mathbb{C}^r$ be a holomorphic embedding. The inclusion $\iota : V \hookrightarrow Y$ factors through the mapping cylinder $M = Y \times \mathbb{C}^r$ as $V \xrightarrow{j} M \xrightarrow{\pi} Y$, where $j = (\iota, \phi)$ is an embedding and π is the projection $(y, z) \mapsto y$. Consider the smooth map $u \circ \pi : M \to X$, which is holomorphic on the submanifold $j(V)$ of M. Assume X satisfies BOPI. Then $u \circ \pi$ can be deformed to a holomorphic map $v : M \to X$, keeping it fixed on $j(V)$. We plan to precompose v by finitely many holomorphic sections of π of the form $\sigma = (\mathrm{id}_Y, \psi) : Y \to M$, where the holomorphic map $\psi : Y \to \mathbb{C}^r$ equals ϕ on Z and approximates ϕ sufficiently well near a subset of Z in a sense to be made precise. Each holomorphic map $v \circ \sigma : Y \to X$ is the identity on the zero section and its derivative restricted to the tangent space of X at a point of the zero section is surjective on the set where ψ approximates ϕ well enough. If these sets cover the zero section, then these maps form a dominating family of sprays for X, showing that X is subelliptic. Then, being Stein, X is in fact elliptic [F1, Lemma 2.2].

We shall use the method of *admissible systems* used by Narasimhan to prove the embedding theorem for Stein spaces and attributed to Grauert in [N]. There are admissible systems $(U_i^\lambda)_{i\in\mathbb{N}}$, $\lambda = 1, \dots, 4\dim X + 1$, that together cover Y. The construction in [N] shows that the relatively compact open subsets U_i^λ of Y may be taken to be Stein. Find compact sets K_i^λ and L_i^λ such that $L_i^\lambda \subset \operatorname{int} K_i^\lambda$, $K_i^\lambda \subset V \cap U_i^\lambda$, and $Z \subset \bigcup_{\lambda,i} L_i^\lambda$. Now fix λ and omit it from the notation. Let $f : V \to \mathbb{C}$ be one of the coordinate functions of ϕ. We define a holomorphic function g on the union $Z \cup \bigcup U_i$ as follows. We set $g = f$ on Z. If $U_i \cap V = \varnothing$, let $g = 0$ on U_i. If U_i intersects V, then we take g on U_i to be an approximation of $f|V \cap U_i$ on K_i with $g = f$ on $Z \cap U_i$. Here we apply the Runge Approximation Theorem for coherent analytic sheaves [Fo, p. 139] to the ideal sheaf $\mathcal{I}_Z$ of Z in the Stein manifold U_i, using the fact that $V \cap U_i$ is Runge in U_i. We first extend $f|Z \cap U_i$ to a holomorphic function $\tilde f$ on U_i; then we approximate $f - \tilde f \in \mathcal{I}_Z(V \cap U_i)$ by $\tilde g \in \mathcal{I}_Z(U_i)$ on K_i and let $g = \tilde f + \tilde g$.

Now we need the following straightforward adaptation of Narasimhan's approximation theorem [N, Theorem 2] to the case of an ideal sheaf.

Approximation Theorem. *Let $(U_i)_{i\in\mathbb{N}}$ be an admissible system on a Stein manifold Y and Z be an analytic subset of Y. Let g be a holomorphic function on $Z \cup \bigcup U_i$. Let $K_i \subset U_i$ be compact and $\epsilon_i > 0$. Then there is a holomorphic function h on Y such that $h = g$ on Z and $|h - g| < \epsilon_i$ on K_i for all i.*

We obtain a holomorphic function h on Y with $h = g = f$ on Z such that h approximates the original function f as closely as we wish on each compact set K_i, so h approximates the derivatives of f as closely as we wish on each compact set L_i. Approximating each coordinate function of ϕ in this way, we obtain a holomorphic map $\psi : Y \to \mathbb{C}^r$ such that the composition $v \circ (\mathrm{id}_Y, \psi) : Y \to X$ is the identity on the zero section and its derivative restricted to the tangent space of X at a point of $Z \cap \bigcup L_i$ is so close to the derivative of u that it is surjective.

We have proved the following result.

Theorem 2. *A Stein manifold satisfying the basic Oka property with interpolation is elliptic.*

Hence, for a Stein manifold, BOPI is equivalent to ellipticity and all Oka-type properties in between, such as subellipticity, POPI, and POPJI. In the argument above, we could avoid taking V to be Runge and simplify the definition of g if we had a detailed proof or a reference to a proof that a Stein manifold can be covered by finitely many admissible systems all of which are subordinate to a given open cover.

Factorization in the projective structure. Turning now from the intermediate structure to the projective structure, we will explicitly factor an arbitrary holomorphic map $f : R \to S$ between Stein manifolds as a projective cofibration into a new mapping cylinder M followed by a pointwise weak equivalence. The new mapping cylinder is constructed

as a (pointwise) pushout

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "i_0"] & S \arrow[d] \\ R \times \Delta^1 \arrow[r] & M \end{tikzcd}$$

in $\mathfrak{S}$, or equivalently as the pushout

$$\begin{tikzcd} R \sqcup R \arrow[r, "f \sqcup \mathrm{id}"] \arrow[d, "i_0 \sqcup i_1"] & S \sqcup R \arrow[d] \\ R \times \Delta^1 \arrow[r] & M \end{tikzcd}$$

Since the left-hand vertical maps are cofibrations in the projective structure, so are the right-hand vertical maps. Since S is cofibrant, the map $R \to S \sqcup R$ that is the identity on R is a cofibration, so the composition $j : R \to S \sqcup R \to M$ is a cofibration. Since R is cofibrant, it follows that M is cofibrant as well. The original map $f : R \to S$ factors as $\pi \circ j$, where π is the obvious projection $M \to S$, which is a left inverse to the map $S \to M$ given directly by the construction of M. Now $R \to R \times \Delta^1$ is a pointwise acyclic cofibration, so its pushout $S \to M$ is pointwise acyclic. Hence, the left inverse π is pointwise acyclic as well. It follows that j is acyclic if and only if f is acyclic, that is, f is a topological homotopy equivalence.

A map in $\mathfrak{S}$ from M to a manifold X is easy to understand explicitly. It is nothing but a pair of maps, one from $\hom_{\mathfrak{S}}(S, X) = \mathcal{O}(S, X)$ and another from

$$\hom_{\mathfrak{S}}(R \times \Delta^1, X) = \hom_{s\mathbf{Set}}(\Delta^1, \mathrm{Hom}_{\mathfrak{S}}(R, X)) = s\mathcal{O}(R, X)_1 = \mathcal{C}(I, \mathcal{O}(R, X))$$

(here, $s\mathbf{Set}$ denotes the category of simplicial sets), which agree when the first map is precomposed with $f : R \to S$ and the second is restricted to $\{0\} \subset I$ (the pushout that defines M is taken levelwise as well as pointwise). In other words, a map $M \to X$ is nothing but a continuous map from the ordinary topological mapping cylinder $S \cup_f (R \times I)$ to X which is holomorphic on each slice S and $R \times \{t\}$, $t \in I$. To put it a third way, a map $M \to X$ is simply a continuous deformation of a holomorphic map $R \to X$ through such maps to a map that factors through f.

An application. If f is a topological weak equivalence, so j is an acyclic cofibration, and X is weakly Oka, it follows that the map $\mathrm{Hom}(M, X) \to \mathrm{Hom}(R, X) = s\mathcal{O}(R, X)$ induced by j is an acyclic fibration and hence surjective on vertices. This means that every holomorphic map $R \to X$ can be continuously deformed through such maps to a map that factors through f by a holomorphic map $S \to X$. (This could for instance be applied to zero-free holomorphic functions, taking $X = \mathbb{C} \setminus \{0\}$, and to meromorphic functions without indeterminacies, taking X to be the Riemann sphere.) The same holds if f is arbitrary and X is weakly Oka and topologically contractible.

The topological mapping cylinder. Let us finally apply the ordinary topological mapping cylinder M of the inclusion of an open Stein subset R in a Stein manifold S, defined as the subset of $S \times [0,1]$ of pairs (x,t) for which $x \in R$ if $t > 0$. The inclusion $R \hookrightarrow S$ factors through M as the inclusion $R \hookrightarrow M$, $x \mapsto (x,1)$, which is a topological cofibration (in the stronger of the two senses used by topologists, namely the sense that goes with Serre fibrations rather than Hurewicz fibrations), followed by the projection $M \to S$, $(x,t) \mapsto x$, which is a topological homotopy equivalence.

Let X be a weakly Oka manifold. Recall that this means that the inclusion $\mathcal{O}(T,X) \hookrightarrow \mathcal{C}(T,X)$ is a weak equivalence in the compact-open topology for every Stein manifold T. Then we have the following diagram.

$$\begin{tikzcd}
\mathcal{O}(S,X) \arrow[dd, "\text{w.eq.}"] \arrow[dr, dotted, "\text{w.eq.}"'] \arrow[drr] & & \\
& \text{pullback} \arrow[r] \arrow[d, "\text{w.eq.}"] & \mathcal{O}(R,X) \arrow[d, "\text{w.eq}"] \\
\mathcal{C}(S,X) \arrow[r, "\text{w.eq.}"] & \mathcal{C}(M,X) \arrow[r, "\text{fibr.}"] & \mathcal{C}(R,X)
\end{tikzcd}$$

The left-hand and right-hand vertical maps are weak equivalences because X is weakly Oka. The right-hand bottom map is a fibration because $R \hookrightarrow M$ is a cofibration. The middle vertical map is a weak equivalence because it is the pullback of a weak equivalence by a fibration. The left-hand bottom map is a weak equivalence because $M \to S$ is a homotopy equivalence. Finally, we conclude that the dotted map

$$\mathcal{O}(S,X) \to \mathcal{C}(M,X) \times_{\mathcal{C}(R,X)} \mathcal{O}(R,X)$$

is a weak equivalence.

One more implication between Oka properties. The dotted map being a weak equivalence implies that a continuous map $S \to X$ which is holomorphic on R can be deformed through continuous maps $M \to X$ which are holomorphic on $R \subset M$ to a holomorphic map $S \to X$. Take open subsets U and V of R with $\overline{U} \subset V$ and $\overline{V} \subset R$. Precomposing the deformation by a section of the projection $M \to S$ of the form $x \mapsto (x, \rho(x))$, where $\rho : S \to [0,1]$ is continuous, $\rho = 1$ on U, and $\rho = 0$ outside V, shows that a continuous map $S \to X$ which is holomorphic on R can be deformed through continuous maps $S \to X$ which are holomorphic on U to a holomorphic map $S \to X$.

Thus wanting to keep a continuous map $S \to X$ holomorphic on a Stein open set where it happens to be holomorphic to begin with, as it is deformed to a holomorphic map, does not lead to a new version of the Oka property, at least as long as we do not mind slightly shrinking the open subset. (This is in the same spirit as our earlier result that POPI subsumes POPA.) It is not known if shrinking is actually necessary.

Remarks. Our results so far clarify and simplify the spectrum of Oka properties and their significance. It is beginning to seem plausible that there is essentially only one Oka property, the one called *the* Oka property in [L2], and that other Oka properties should be viewed as variants or special cases of it. (Although they are technically more difficult to work with, the parametric properties are more interesting than the basic ones from a homotopy-theoretic point of view: it is more natural to work with weak equivalences of mapping spaces than maps that merely induce surjections of path components.) It also appears that there is no reason to look for intermediate model structures of relevance to Oka theory beyond the one already defined in [L2].

Furthermore, the chief practical sufficient condition for the Oka property, ellipticity, turns out to be equivalent to it, at least for Stein manifolds. To what extent this generalizes to arbitrary manifolds is a most interesting open question. If X is any manifold satisfying BOPI (or just BOPA) and $p \in X$, then there is a holomorphic map from Euclidean space of dimension $\dim_p X$ to X that takes the origin to p and is a submersion there. The question is whether such maps can be made to fit together to make a dominating spray or a dominating family of sprays or some weaker geometric structure that could still be used to establish the Oka property. In the remainder of the paper we take a step towards answering this question in the affirmative.

Cofibrant models. A cofibrant model for a complex manifold X in the intermediate model structure on $\mathfrak{S}$ is an acyclic intermediate fibration from an intermediately cofibrant prestack to X, or, in other words, a factorization of the map from the empty manifold to X of the same sort as before. (There is a distinction between the empty prestack (the initial object in $\mathfrak{S}$) and the prestack represented by the empty manifold, but here it is immaterial.) Such a factorization always exists by the axioms for a model category, but we seek a reasonably explicit cofibrant model represented by a manifold S. An intermediately cofibrant manifold is Stein — we defer the rather technical proof of this to the end of the paper — so S will be Stein and the acyclic intermediate fibration $S \to X$ holomorphic.

Our motivation for seeking cofibrant models represented by Stein manifolds is threefold. First, since fibrancy passes up and down in an acyclic fibration, a Stein cofibrant model allows us to reduce the problem of geometrically characterizing the Oka property to the Stein case, of which we have already disposed. Second, an acyclic intermediate fibration from a Stein manifold to X is weakly final among maps from Stein manifolds into X: all such maps factor through it (not necessarily uniquely). This is a nontrivial and apparently new notion that may be of independent complex-analytic interest. Third, from a homotopy-theoretic point of view, we are taking explicit factorizations beyond the relatively easy case in which the source and target are both cofibrant.

The Jouanolou Trick. The so-called Jouanolou Trick, invented for the purpose of extending K-theory from affine schemes to quasi-projective ones [J], is the observation that every quasi-projective scheme X carries an affine bundle whose total space A is affine. Then X and A have the same motive. This can be used to reduce various questions in algebraic geometry to the affine case. Of interest here is the expectation that A is a cofi-

brant model for X in the intermediate structure. Let us outline an analytic version of the Jouanolou Trick, starting with projective space.

Let $\mathbb{P}_n$ denote n-dimensional complex projective space. Let $\mathbb{Q}_n$ be the complement in $\mathbb{P}_n \times \mathbb{P}_n$ of the hypersurface of points $([z_0 : \cdots : z_n], [w_0 : \cdots : w_n])$ with $z_0 w_0 + \cdots + z_n w_n = 0$. This hypersurface is the preimage of a hyperplane by the Segré embedding $\mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_{n^2+2n}$, so $\mathbb{Q}_n$ is Stein. Let π be the projection $\mathbb{Q}_n \to \mathbb{P}_n$ onto the first component. It is easily seen that π has the structure of an affine bundle with fibre $\mathbb{C}^n$ (of course without holomorphic sections). In particular, π is an acyclic elliptic bundle and hence an acyclic subelliptic submersive Serre fibration (SSSF). Our conjecture from [L2] that an SSSF is an intermediate fibration remains open, so for now π is only a candidate for a cofibrant model of $\mathbb{P}_n$. We shall continue with SSSFs standing in for intermediate fibrations.

Towards the general case. Choose a class $\mathcal{G}$ of acyclic SSSFs, containing all affine bundles and closed under taking pullbacks by arbitrary holomorphic maps. Call the maps in $\mathcal{G}$ *good* maps. For example, $\mathcal{G}$ could be the class of affine bundles; holomorphic fibre bundles with a contractible subelliptic fibre, such as $\mathbb{C}^k$ for some $k \geq 0$; locally smoothly or real-analytically trivial acyclic subelliptic submersions; or all acyclic SSSFs. Call a complex manifold *good* if it is the target (and hence the image) of a good map from a Stein manifold.

We have seen that projective spaces are good and so are Stein manifolds, obviously. The pullback of a good map $g : S \to Y$ by any holomorphic map $f : X \to Y$ is a good map $f^*g : R \to X$ (R is smooth, that is, a manifold, because g is a submersion). If f is a covering map, a finite branched covering map, the inclusion of a (closed) submanifold, or the inclusion of the complement of an analytic hypersurface, then so is the pullback map $g^*f : R \to S$, so if S is Stein, then R is also Stein. By blowing up the complement of a Zariski-open subset of a projective variety, turning it into a hypersurface, we see that quasi-projective manifolds are good. Finally, it is easy to see that the product of good manifolds is good.

The class of good manifold thus appears to be quite large (even with the smallest possible $\mathcal{G}$). It contains all Stein manifolds and quasi-projective manifolds and is closed under taking products, covering spaces, finite branched covering spaces, submanifolds, and complements of analytic hypersurfaces. We do not know if every manifold, or even every domain in Euclidean space, is the target of an acyclic SSSF from a Stein manifold.

An acyclic SSSF from a Stein manifold S to X is weakly final among holomorphic maps from Stein manifolds to X. Namely, if R is Stein and $R \to X$ is a holomorphic map, then there is a continuous lifting $h : R \to S$ since $S \to X$ is an acyclic Serre fibration, and because $S \to X$ is a subelliptic submersion, h can be deformed to a holomorphic lifting. (This can of course be strengthened to include interpolation on a submanifold of R.) We do not know if every manifold has a weakly final map from a Stein manifold.

The next theorem is a weak variant of one of the interesting statements that follow from the conjecture that an SSSF is an intermediate fibration.

Theorem 3. *Let X and Y be complex manifolds and $f : X \to Y$ be a submersive subelliptic Serre fibration. If Y has the basic Oka property with interpolation, then so does X. If f is acyclic and X has the basic Oka property with interpolation, then so does Y.*

This statement with BOPI replaced by POPI is a direct consequence of our conjecture since POPI is equivalent to intermediate fibrancy [L2], but BOPI is all we can handle at present. Forstnerič has proved an analogous result for BOPA [F3]. (He calls BOPA *the Oka property.*) Interestingly enough, he does not require acyclicity in order to pass from the source to the target. We do not know if this holds for BOPI. It is also proved in [F3] that POPA passes up in an SSSF.

Proof. First suppose Y satisfies BOPI. Let $h : S \to X$ be a continuous map from a Stein manifold S such that the restriction of h to a submanifold T of S is holomorphic. We need to show that h can be deformed to a holomorphic map $S \to X$ keeping it fixed on T.

By assumption, the composition $f \circ h$ can be deformed to a holomorphic map $g : S \to Y$ keeping it fixed on T. By the topological Axiom SM7, since f is a Serre fibration, the map

$$\mathcal{C}(S, X) \to \mathcal{C}(S, Y) \times_{\mathcal{C}(T,Y)} \mathcal{C}(T, X)$$

is a fibration, so as $f \circ h$ is deformed to g, we can deform h with it to a continuous map $k : S \to X$, keeping it fixed on T. This yields the following square of holomorphic maps.

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S \arrow[r, "g"'] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

Since f is a subelliptic submersion, we can deform k to a holomorphic lifting in this square. Thus, in two steps, h has been deformed to a holomorphic map, keeping it fixed on T.

Second, suppose f is acyclic and X satisfies BOPI. Let $h : S \to Y$ be a continuous map from a Stein manifold S such that the restriction of h to a submanifold T of S is holomorphic. Since f is an acyclic Serre fibration, there is a continuous lifting $T \to X$ of $h|T$. Since f is a subelliptic submersion, this lifting can be deformed to a holomorphic lifting. Again, since f is an acyclic Serre fibration, there is a continuous lifting in the square

$$\begin{tikzcd} T \arrow[r, "\text{hol.}"] \arrow[d] & X \arrow[d, "f"] \\ S \arrow[r, "h"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since X satisfies BOPI, the lifting can be deformed to a holomorphic map, keeping it fixed on T. Postcomposing with f gives a deformation of h to a holomorphic map, keeping it fixed on T, and the proof is complete. $\square$

Taking T to be empty proves the theorem with BOPI replaced by BOP. For BOP, acyclicity is required in order to pass from the source to the target, as shown by the simple example of the universal covering map from the disc to the punctured disc.

A holomorphic intermediate fibration satisfies the conclusion of Theorem 3, because, as shown in [L2], it has the properties of an SSSF that were used in the proof above.

It is an open question whether a manifold X fibred over an elliptic manifold by elliptic manifolds is elliptic. It follows from Theorems 2 and 3 that the answer is affirmative if X is Stein; in general X at least satisfies BOPI by Theorem 3.

Here is our geometric characterization of the basic Oka property with interpolation for an arbitrary good manifold, extending the characterization by ellipticity for Stein manifolds given by Theorem 2.

Theorem 4. *A good manifold has the basic Oka property with interpolation if and only if it is the image of a good map from an elliptic Stein manifold.*

Proof. Let X be a manifold and $S \to X$ be a good map. If S is elliptic, then S satisfies BOPI, so X does too by Theorem 3. (For this direction, we do not need S to be Stein.) Conversely, suppose X satisfies BOPI and is good, so there is a good map $S \to X$ with S Stein. Then S satisfies BOPI by Theorem 3 and is therefore elliptic by Theorem 2. $\square$

We can state this more explicitly for quasi-projective manifolds.

Theorem 5. *A quasi-projective manifold has the basic Oka property with interpolation if and only if it carries an affine bundle whose total space is elliptic and Stein.*

Note that if good maps are intermediate fibrations, as predicted by our conjecture, so POPI passes down in a good map, then it follows that BOPI and POPI are equivalent for good manifolds.

A converse Oka Principle. By the definition of the intermediate model structure, a Stein manifold is intermediately cofibrant. We conclude this paper by proving that, conversely, an intermediately cofibrant complex manifold is Stein. It is then immediate that projective cofibrancy is also equivalent to being Stein. This may be viewed as a "converse Oka Principle" for manifolds. Characterizing the Stein property as cofibrancy complements the characterizations in [L2] of the parametric Oka property with and without interpolation as intermediate and projective fibrancy, respectively. We hope to treat the general case of holomorphic maps later.

Remark on fullness. Before proceeding to the proof, we recall from [L2] that while the Yoneda embedding of the category $\mathcal{S}$ of Stein manifolds into the category $\mathfrak{S}$ of prestacks on $\mathcal{S}$ is of course full (even simplicially full), we have been unable to determine whether its extension to an embedding of the category $\mathcal{M}$ of all complex manifolds into $\mathfrak{S}$, taking a manifold Z to the prestack $s\mathcal{O}(\cdot, Z)$, is full. It is easy to demonstrate a weaker fullness property, which is required here and suffices for many purposes. A morphism $\eta : X \to Y$ in $\mathfrak{S}$ between complex manifolds (that is, between the prestacks represented by them) is a

natural transformation $s\mathcal{O}(\cdot, X) \to s\mathcal{O}(\cdot, Y)$ of contravariant simplicial functors $\mathcal{S} \to s\mathbf{Set}$. Considering what this means, see e.g. the definitions in [GJ, Chapter IX], we observe that *η restricted to the vertex level* is nothing but a morphism $\mathcal{O}(\cdot, X) \to \mathcal{O}(\cdot, Y)$ in the category $\operatorname{Pre}\mathcal{S}$ of presheaves of sets on $\mathcal{S}$. Such a morphism is induced by a holomorphic map $X \to Y$ since by [L1, Proposition 4.2], the functor $\mathcal{M} \to \operatorname{Pre}\mathcal{S}$ taking a manifold Z to $\mathcal{O}(\cdot, Z)$ is a full embedding. What we do not know unless X is Stein is whether this holomorphic map also induces the higher-level maps in η. For all we know, distinct morphisms $X \to Y$ in $\mathfrak{S}$ might coincide at the vertex level.

We conclude that if we restrict any diagram of complex manifolds and morphisms in $\mathfrak{S}$ to the vertex level, the morphisms will be induced by holomorphic maps, so we get a diagram in $\mathcal{M}$. In other words, restriction to the vertex level gives a retraction functor from the full subcategory of $\mathfrak{S}$ generated by $\mathcal{M}$ onto $\mathcal{M}$ itself.

Theorem 6. *Let X be a complex manifold. The following are equivalent.*

(1) *X is intermediately cofibrant.*
(2) *X is projectively cofibrant.*
(3) *X is Stein.*

Proof. By the definition of the intermediate and projective model structures on $\mathfrak{S}$, it is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Assuming that X is intermediately cofibrant, we need to prove that X is Stein. It suffices to show that we can prescribe the values of a holomorphic function on X on any two-point subset of X (this gives holomorphic separability) and on some infinite subset of any infinite discrete subset of X (this gives holomorphic convexity). By definition of the intermediate model structure, X is a retract $X \to Y \to X$ of a cell complex Y, that is, a prestack which is the target of a transfinite composition with source $\varnothing$ of pushouts of generating cofibrations

$$S \times \partial\Delta^n \cup_{T\times\partial\Delta^n} T \times \Delta^n \to S \times \Delta^n,$$

where $T \hookrightarrow S$ is a Stein inclusion and $n \geq 0$. More explicitly, $Y = \operatorname{colim}_{\alpha<\lambda} A_\alpha$, where $A : \lambda \to \mathfrak{S}$ is a functor from an ordinal λ with $A_0 = \varnothing$ such that for every limit ordinal $\gamma < \lambda$, the induced map $\operatorname{colim}_{\alpha<\gamma} A_\alpha \to A_\gamma$ is an isomorphism, and such that for every successor ordinal $\alpha < \lambda$, the map $A_{\alpha-1} \to A_\alpha$ is a pushout of a generating cofibration given by a Stein inclusion $T_\alpha \hookrightarrow S_\alpha$ and an integer $n_\alpha \geq 0$.

Let E be an infinite discrete subset of X, viewed as a discrete Stein manifold. Consider the subset of λ of ordinals α such that the restriction of $E \hookrightarrow X \to Y$ to an infinite subset of E factors through $S_\alpha \times \Delta^{n_\alpha}$. This subset is not empty: since colimits in $\mathfrak{S}$ are taken levelwise and pointwise, the map $E \to Y$ itself, as a vertex of $Y(E)$, is the image of a vertex of $(S_\alpha \times \Delta^{n_\alpha})(E)$ for some α. Let β be the smallest element of this subset and let E' be an infinite subset of E such that the restriction $E' \to Y$ factors through $S_\beta \times \Delta^{n_\beta}$. At the vertex level, generating cofibrations with $n \geq 1$ are isomorphisms since $\partial\Delta^n_0 = \{0, \ldots, n\} = \Delta^n_0$, so $n_\beta = 0$, for otherwise $E' \to Y$ would factor through $A_{\beta-1}$

and hence through $S_\alpha \times \Delta^{n_\alpha}$ for some $\alpha < \beta$. Since $E' \to Y$ is a monomorphism, so is $E' \to S_\beta$. The composition $E' \to S_\beta \to Y \to X$, which is simply the inclusion of E' in X, has discrete image, and the morphism $S_\beta \to X$ is induced by a holomorphic map, so $E' \to S_\beta$ has discrete image and we can view E' as a discrete subset of S_β. The restriction of $E \to Y$ to $E' \cap T_\beta$ factors through $A_{\beta-1}$ and hence through $S_\alpha \times \Delta^{n_\alpha}$ for some $\alpha < \beta$, so $E' \cap T_\beta$ is finite. Let F be the infinite subset $E' \setminus T_\beta$ of E.

Similarly, if F is a two-point subset of X, let β be the smallest element of λ such that $F \hookrightarrow X \to Y$ factors through $S_\beta \times \Delta^{n_\beta}$. Then $n_\beta = 0$ and T_β cannot contain F.

In either of the two cases, take a map $F \to \mathbb{C}$ and extend it to $F \cup T_\beta$ by mapping T_β to a constant. Map $A_{\beta-1}$ to the same constant, say through the map to the final prestack. Since $F \cup T_\beta$ is a submanifold of the Stein manifold S_β, the holomorphic map $F \cup T_\beta \to \mathbb{C}$ has a holomorphic extension to S_β.

$$\begin{tikzcd}
 & & F \arrow[dl] \arrow[d] \arrow[dr] & \\
T_\beta \arrow[r] \arrow[d] & S_\beta \arrow[r] \arrow[d] & \mathbb{C} & X \arrow[d] \\
A_{\beta-1} \arrow[r] \arrow[urr] & A_\beta \arrow[rr] \arrow[ur, dotted] & & Y \arrow[ul, dotted]
\end{tikzcd}$$

By the pushout property of A_β, we get an induced map $A_\beta \to \mathbb{C}$. Since $A_\beta \to Y$ is an intermediate cofibration and $\mathbb{C}$ is intermediately fibrant and acyclic, there is an extension $Y \to \mathbb{C}$. Finally, the restriction to the vertex level of the composition $X \to Y \to \mathbb{C}$ is a holomorphic map by our fullness remark and it extends the given map on F. $\square$

References

[Fo] O. Forster, *Einführung in die Komplexe Analysis Mehrerer Veränderlichen*, Vorlesungen an der Universität Regensburg, 1973.

[F1] F. Forstnerič, *The Oka principle for sections of subelliptic submersions*, Math. Z. **241** (2002), 527–551.

[F2] ———, *The homotopy principle in complex analysis: a survey*, Explorations in Complex and Riemannian Geometry: A Volume Dedicated to Robert E. Greene, Contemporary Mathematics 332, Amer. Math. Soc., 2003, pp. 73–99.

[F3] ———, *Convex approximation implies Oka's property*, preprint, available on the ArXiv (2004).

[FP] ——— and J. Prezelj, *Extending holomorphic sections from complex subvarieties*, Math. Z. **236** (2001), 43–68.

[GJ] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser Verlag, 1999.

[G] M. Gromov, *Oka's principle for holomorphic sections of elliptic bundles*, Jour. Amer. Math. Soc. **2** (1989), 851–897.

[J] J. P. Jouanolou, *Une suite exacte de Mayer-Vietoris en K-théorie algébrique*, Algebraic K-theory I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math. 341, Springer Verlag, 1973, pp. 293–316.

[KK] L. Kaup and B. Kaup, *Holomorphic functions of several variables*, de Gruyter Studies in Mathematics 3, Walter de Gruyter, 1983.

[L1] F. Lárusson, *Excision for simplicial sheaves on the Stein site and Gromov's Oka principle*, Internat. J. Math. **14** (2003), 191–209.

[L2] F. Lárusson, *Model structures and the Oka Principle*, J. Pure Appl. Algebra **192** (2004), 203–223.

[N] R. Narasimhan, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math. **82** (1960), 917–934.

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